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Stability of the unsteady viscous flow in a curved pipe

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The linear stability of the flow of an incompressible viscous fluid through a curved pipe of circular cross-section is considered. There is a sinusoidal pressure gradient, with zero mean, acting down the pipe. The flow is shown to be unstable to a Taylor–Görtler mode of instability, with vortices aligned with the basic flow first appearing at the outer bend of the pipe when a critical value of the Taylor number is exceeded. A WKBJ perturbation solution is constructed and the form of the vortex amplitude is determined. This solution is found to break down in the vicinity of the pipe's outer bend, and an inner solution is presented to overcome this. The solution is determined by identifying a saddle point in the complex plane of the cross-sectional angle coordinate. This leads to an eigenvalue problem for the Taylor number, for fixed wavenumber and cross-sectional angle coordinate, which in turn leads to the determination of the critical Taylor number above which instability sets in.

1. Introduction

The stability of periodic laminar flows forms an important part of fluid dynamics, both from a mathematical and a physical point of view. Such flows occur frequently in nature and one such example, related to this work, is blood flow in large arteries.

Periodic laminar flows can be divided into those modulated about some non-zero mean, and purely oscillatory ones, depending on the problem at hand. The methods used to analyse such flows are also different. For instance, when modulation is present, the instability is often associated with the mean flow, and the stability parameters depend on the unperturbed flow characteristics. This means that, in many cases, perturbation methods can be used to describe the instability. Some examples can be found in Grosch & Salwen (1968), Hall (1975*a*) (where the stability of modulated plane Poiseuille flow is considered) and Hall (1975*b*), Riley & Lawrence (1976) (where the stability of modulated circular Couette flow is described by asymptotic and numerical methods, respectively). For a review of the stability of periodic flows see Davis (1976).

In the case of purely oscillatory flows, however, perturbation methods can no longer be applied in general, and numerical solutions are usually needed to resolve the stability problem. Rosenblat (1968) studied the instability of purely oscillatory cylinder flows, set up by the motion of an incompressible inviscid fluid between concentric infinite cylinders. Instability is found which is associated with a phase lag between velocity and vorticity. With viscosity present, however, Stokes layers are formed at solid surfaces, and an understanding of their stability mechanisms is needed. Stokes layers can be found, for example, at the pipe walls in the problem studied here, at the boundary of a cylindrical body oscillating along a diameter

(Stuart 1966; Riley 1967), and at the bottom of a channel over which a gravity wave is propagating (Longuet-Higgins 1953).

Stability properties of Stokes layers depend on their local geometry. For example, different mechanisms, such as centrifugal effects, come into play in a curved Stokes layer, but not in a flat one. The stability of the Stokes layers on the walls of the flow between parallel plates when one of the plates oscillates harmonically in time, has been calculated by von Kerczek & Davis (1974). Hall (1978) analysed flat Stokes layer stability both in the absence and presence of an upper rigid boundary. Von Kerczek & Davis find linear stability (for evolution of disturbances over a cycle) for Reynolds numbers, based on Stokes layer thickness, of up to about 800. It is also conjectured that the flow remains linearly stable at higher Reynolds numbers. Tromans (1977) and Cowley (1986) have shown by a quasi-steady analysis that for large Reynolds numbers, Stokes layers are locally unstable to Rayleigh modes (inflexional instabilities). Some experiments are cited by Cowley in support of this mechanism. It is interesting to note, however, that for Stokes layers which include centrifugal or stratification effects, Floquet-theory stability analysis is in very good agreement with experiments (see Seminara & Hall (1976) and von Kerczek & Davis (1976) respectively). An explanation for this is presented by Cowley (1986) for the problem of Seminara & Hall.

Our concern is with flows in curved geometries that exhibit centrifugal instability. Of interest, therefore, is the work of Seminara & Hall cited above, where they investigate the linear stability of the Stokes layer found on an infinite cylinder that oscillates harmonically about its axis in an unbounded viscous fluid. The streamlines inside the Stokes layer are curved, and instability sets in as azimuthal Taylor vortices which are periodic in the axial direction. It should be noted that the undisturbed flow in the above problem has a component in the azimuthal direction alone and is a function of the radial distance from the axis. Hall (1984) investigated the boundary-layer stability on a transversely oscillating cylinder. The basic flow now has components in two spatial directions (no component parallel to the cylinder axis). The flow is shown to be locally unstable to Taylor vortices, which form at positions where the Stokes layers are parallel to the direction of motion of the cylinder. It is further conjectured that such instabilities can be found in more complicated streaming flows and the analysis is extended to the motion of an elliptical cylinder.

The study undertaken here is more complicated than previous ones, in that the underlying flow is three-dimensional. It comprises the main motion down the pipe and a two-dimensional secondary flow in the pipe cross-section. This property makes it hard to classify the instability as either of the Taylor- or of the Görtler-vortex type, but some comparisons can be made towards an understanding of the physical mechanisms involved.

The structure of the flow field near the walls can be of considerable practical importance. If, for instance, our model is taken to represent blood flow in large arteries, then the instability can affect the shear-stress distribution at the walls and hence the uptake of lipoproteins, deposition on the walls and perhaps the onset of atheroma.

A detailed asymptotic analysis of the fully developed unperturbed flow in a curved pipe, under the action of a sinusoidal pressure gradient, was first given by Lyne (1971). It is assumed that δ, the ratio of the pipe radius to its curvature, is small, and the two parameters

$$\epsilon = \frac{W}{a\omega}\left(\frac{a}{R}\right)^{\frac{1}{2}}, \quad R_{\mathrm{s}} = \frac{W^2}{R\omega}\frac{a}{\nu},$$

are found for the problem. Here W is a typical velocity down the pipe, ω is the frequency of oscillation of the basic flow, a the radius of the pipe, R its radius of curvature and ν the kinematic viscosity of the fluid. The parameter ϵ^2 represents the ratio of the square of the oscillation amplitude of particles in the core down the pipe, to the product of the pipe radius and its radius of curvature. This parameter is taken to be small in the regime studied here. R_s is the Reynolds number for the secondary flow based on the pipe radius. Another important parameter is β, given by

$$\beta^2 = \frac{2\nu}{a^2\omega} = \frac{2\epsilon^2}{R_s}.$$

Physically β represents the ratio of the Stokes-layer thickness $(2\nu/\omega)^{\frac{1}{2}}$ to the pipe radius. Lyne's analysis depends on β being small, which implies that viscous effects are confined to a thin layer on the wall. The influence of the parameter β on biological flows was recognized by Womersley (1955).

We are concerned with the evolution of disturbances inside the Stokes layer, where the curvature of the streamlines is of order R. Thus, a Taylor number Ta can be defined in the usual way,

$$Ta = \frac{W^2\left[\left(\dfrac{2\nu}{\omega}\right)^{\frac{1}{2}}\right]^3}{R\nu^2} = 2\frac{W^2}{R\omega}\frac{a}{\nu}\left(\frac{2\nu}{a^2\omega}\right)^{\frac{1}{2}} = 2R_s\beta.$$

Here we require that Ta is an order-one parameter, in order to produce centrifugal instability of the Taylor-vortex type. We choose $R_s = 2T\beta^{-1}$ ($Ta = 4T$ now), in which case the problem is reduced to depend on one small parameter β and an order-one parameter T, by the substitution $\epsilon^2 = \beta T$ which follows from above.

We note here, that the Taylor number for the secondary flow is proportional to $(W^2/R\omega)^2(\nu/\omega)^{\frac{3}{2}}/a\nu^2 = \beta Ta^2/\surd 2$, and is therefore much smaller than the Taylor number for the axial flow. Physically, therefore, we anticipate that vortices will be formed aligned with the flow down the pipe and with characteristic wavenumbers scaled on the Stokes-layer thickness.

It follows that in the construction of asymptotic solutions two lengthscales, $O(1)$ and $O(\beta)$, become important in the cross-section plane. The perturbation solutions can be expanded in powers of β and a WKBJ method can be used. Such a procedure was adopted by Walton (1978) in his stability investigation of the steady flow in a narrow spherical annulus. He found that the solution becomes singular near the equator, which suggests the need for an inner expansion. Some difficulties arise, however, in connection with the inner-solution behaviour away from the equator. Soward & Jones (1983) resolved the problem by identification of the value of T for which the inner solutions behave correctly away from the equator. This method is modified here for the more complicated unsteady basic flow at hand.

The procedure adopted in the rest of the paper is as follows. In §2 the unperturbed flow, and in particular that inside the Stokes layer, is described. In §3 the linear-stability problem is formulated and in §4 an analytical solution in the neighbourhood of the outer bend (the analysis takes the same form at the inner bend) is obtained as a power series in β. This solution is found to break down in the immediate vicinity of the outer bend. In §5 an inner solution is described by identification of the correct Taylor number for the flow. In §6 a numerical solution is presented that calculates the neutral stability curve of T, and in §7 we give the results and make some comments.

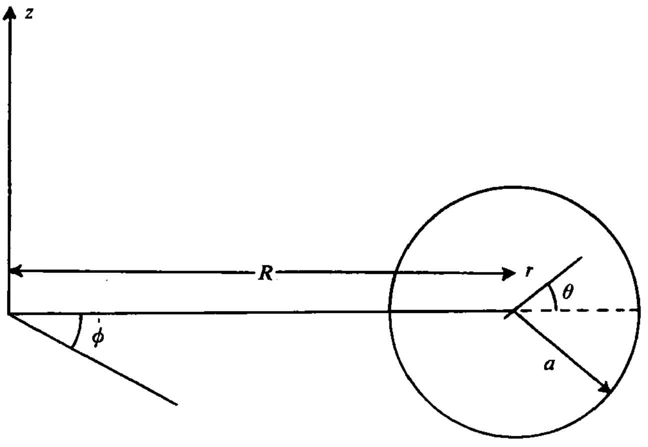

FIGURE 1. The coordinate system.

2. The basic flow

The velocity vector $\boldsymbol{u}$ is taken to have components (u, v, w) which correspond to the spatial coordinates (r, θ, ϕ), where r and θ are polar coordinates in the pipe cross-section and $R\phi$ denotes distance down the pipe; $\boldsymbol{u}$ is assumed to be independent of ϕ when the basic flow is fully developed. The coordinate system is shown in figure 1.

If the Navier–Stokes equations are considered in the above coordinate system, together with a sinusoidal pressure gradient in the ϕ-direction of the form

$$-\frac{\partial}{\partial\phi}\frac{p}{\rho} = RW\omega\cos\omega t,$$

then a balance of viscous and pressure terms implies that there exists a layer of thickness $O(2\nu/\omega)^{\frac{1}{2}}$ where viscous effects are dominant, with a potential flow in the core. Similar balances in the r- and θ-momentum equations yield that inside the viscous layer, $v = O(W^2/R\omega)$ and $u = O(W^2\beta/R\omega)$. Following Lyne we non-dimensionalize u and v with respect to $W^2/R\omega$ and w with respect to W. The radial distance is non-dimensionalized with respect to a, the time t with respect to $\omega^{-1}(t = \omega\tau)$ and the pressure with respect to $\rho(a/R)\,W^2$. The system of equations obtained gives a general description of the viscous flow both near the wall and in the core of the pipe. These equations are

$$u_\tau + \epsilon^2\left(uu_r + \frac{1}{r}vu_\theta - \frac{1}{r}v^2\right) - w^2\cos\theta = -p_r - \tfrac{1}{2}\beta^2\frac{1}{r}\frac{\partial}{\partial\theta}\left(v_r + \frac{1}{r}v - \frac{1}{r}u_\theta\right), \tag{2.1a}$$

$$v_\tau + \epsilon^2\left(uv_r + \frac{1}{r}vv_\theta + \frac{1}{r}uv\right) + w^2\sin\theta = \frac{1}{r}p_\theta + \tfrac{1}{2}\beta^2\frac{\partial}{\partial r}\left(v_r + \frac{1}{r}v - \frac{1}{r}u_\theta\right), \tag{2.1b}$$

$$w_\tau + \epsilon^2\left(uw_r + \frac{1}{r}vw_\theta\right) = \cos\tau + \tfrac{1}{2}\beta^2\left(w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta}\right), \tag{2.1c}$$

$$u_r + \frac{1}{r}u + \frac{1}{r}v_\theta = 0. \tag{2.1d}$$

It can be seen that the continuity equation (2.1 d) is satisfied by the introduction of a stream function ψ, with the velocities u, v given by

$$u = \frac{1}{r}\psi_\theta, \quad v = -\psi_r.$$

We are interested in the appearance of instability near the walls, and more specifically inside the Stokes layer which has thickness $(2\nu/\omega)^{\frac{1}{2}}\beta$. A scaling of r and ψ inside the Stokes layer is therefore necessary; these scalings are

$$\eta = \beta^{-1}(1-r), \quad \Psi = \beta^{-1}\psi,$$

where the new variables η, Ψ are the radial coordinate and stream function, respectively, inside the Stokes layer. The solution for the basic flow inside the Stokes layer is given by Lyne as a series expansion in β. This is

$$\Psi = \Psi_0 + \beta\Psi_1 + \beta^2\Psi_2 + \ldots, \tag{2.2a}$$

$$w = w_{B0} + \beta w_{B1} + \ldots. \tag{2.2b}$$

In (2.2a, b) above, $\Psi_0, \Psi_1, \ldots, w_{B0}, \ldots$, are functions of τ, η and θ which must be calculated for fixed values of the secondary Reynolds number R_s. In particular Lyne obtains asymptotic solutions as $\beta \to 0$ for both $R_s \to 0$ and $R_s \to \infty$. As was concluded in the introduction, for the type of centrifugal instability considered here, the latter limit must be taken (in fact the limit $R_s = 2T\beta^{-1}$ is needed as $\beta \to 0$). The solutions for Ψ_0, Ψ_1 as $R_s \to \infty$ are (see Lyne 1971),

$$\Psi_0 = f(\eta, \tau)\sin\theta,$$

where

$$\begin{aligned} f(\eta,\tau) = \{&-\tfrac{1}{4}\eta + \tfrac{5}{8} - \tfrac{1}{8}\,e^{-2\eta} - \frac{1}{\sqrt{2}}\,e^{-\eta}\cos(-\eta+\tfrac{1}{4}\pi) + \tfrac{5}{8}\,e^{-\sqrt{2}\eta}\cos(2\tau-\sqrt{2}\eta+\tfrac{1}{4}\pi) \\ &-\tfrac{1}{16}\sqrt{2}\,e^{-2\eta}\cos(2\tau-2\eta+\tfrac{1}{4}\pi) + \tfrac{5}{8}\,e^{-\sqrt{2}\eta}\cos(2\tau-\sqrt{2}\eta+\tfrac{1}{4}\pi) \\ &-\frac{1}{\sqrt{2}}\,e^{-\eta}\cos(2\tau-\eta+\tfrac{1}{4}\pi) + \tfrac{1}{16}(9\sqrt{2}-10)\cos(2\tau+\tfrac{1}{4}\pi)\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \Psi_1 = \{&-\tfrac{1}{16}\sqrt{2}\eta\,e^{-2\eta}\cos(2\tau-2\eta+\tfrac{1}{4}\pi) - \tfrac{1}{16}\,e^{-2\eta}\cos(2\tau-2\eta) - \tfrac{1}{4}\sqrt{2}\eta \\ &\times e^{-\eta}\cos(2\tau-\eta+\tfrac{1}{4}\pi) + \tfrac{1}{4}\,e^{-\eta}\cos(2\tau-\eta) + \tfrac{5}{16}\,e^{-\sqrt{2}\eta}\cos(2\tau-\sqrt{2}\eta+\tfrac{1}{4}\pi) \\ &-\tfrac{1}{32}(16\sqrt{2}-15)\,e^{-\sqrt{2}\eta}\cos(2\tau-\sqrt{2}\eta) - \tfrac{1}{4}\sqrt{2}\,e^{-\eta}\cos(-\eta+\tfrac{1}{4}\pi) \\ &+\tfrac{1}{4}\,e^{-\eta}\cos\eta + \tfrac{1}{18}\eta\,e^{-2\eta} - \tfrac{1}{16}\,e^{-2\eta} - \tfrac{1}{16}(9\sqrt{2}-10)\eta\cos(2\tau+\tfrac{1}{4}\pi) \\ &+\tfrac{1}{2}h + \tfrac{1}{32}(16\sqrt{2}-21)\cos 2\tau - \tfrac{3}{16}\}\sin\theta + \eta^2 I(\theta). \end{aligned} \tag{2.4}$$

In (2.4) above $I(\theta)$ is a function of θ that is determined by the matching of the Stokes-layer solution with the core flow in the limit $R_s \to \infty$.

The first two terms in the expression for w are

$$w_{B0} = \sin\tau - e^{-\eta}\sin(\tau-\eta), \tag{2.5}$$

$$w_{B1} = -\tfrac{1}{2}\eta\,e^{-\eta}\sin(\tau-\eta). \tag{2.6}$$

Hence if the basic flow is denoted by $\boldsymbol{u}_{\mathrm{B}} = (u_{\mathrm{B}}, v_{\mathrm{B}}, w_{\mathrm{B}})$, we have the following expressions holding:

$$u_{\mathrm{B}} = \beta u_{\mathrm{B0}} + \beta^2 u_{\mathrm{B1}} + \dots,$$

$$v_{\mathrm{B}} = v_{\mathrm{B0}} + \beta v_{\mathrm{B1}} + \dots,$$

where

$$u_{\mathrm{B0}} = \partial \Psi_0/\partial\theta, \quad u_{\mathrm{B1}} = \partial \Psi_1/\partial\theta, \quad v_{\mathrm{B0}} = \partial \Psi_0/\partial\eta, \quad v_{\mathrm{B1}} = \partial \Psi_1/\partial\eta.$$

It should be noted here that the solutions given above are strictly valid in the limit $R_{\mathrm{s}} \to \infty$ with β held fixed. We are interested, however, in the basic flow in the special limit $R_{\mathrm{s}} = 2T\beta^{-1}$ as $\beta \to 0$. This poses no difficulties because the effect of R_{s} is not explicit in the Stokes-layer equations for Ψ_0, Ψ_1, w_{B0}, w_{B1} at least, but affects the flow in the core, where it takes on the role of a conventional Reynolds number. The expressions (2.3)–(2.6) are therefore valid representations of the basic flow as $\beta \to 0$ in the regime of instability under consideration.

3. Formulation of the linear stability problem

Before the stability problem is posed we present the equations of motion inside the Stokes layer. These are equations (2.1) written in terms of the Stokes variable η. Thus,

$$u_\tau + \epsilon^2\left(-\frac{1}{\beta}uu_\eta + vu_\theta - v^2\right) - w^2\cos\theta = \frac{1}{\beta}p_\eta - \tfrac{1}{2}\beta^2\frac{\partial}{\partial\theta}\left(-\frac{1}{\beta}v_\eta + v - u_\theta\right), \quad (3.1a)$$

$$v_\tau + \epsilon^2\left(-\frac{1}{\beta}uv_\eta + vv_\theta + uv\right) + w^2\sin\theta = -p_\theta - \tfrac{1}{2}\beta\frac{\partial}{\partial\eta}\left(-\frac{1}{\beta}v_\eta + v - u_\theta\right), \quad (3.1b)$$

$$w_\tau + \epsilon^2\left(-\frac{1}{\beta}uw_\eta + vw_\theta\right) = \cos\tau + \tfrac{1}{2}\beta^2\left[\frac{1}{\beta^2}w_{\eta\eta} - \frac{1}{\beta}w_\eta + w_{\theta\theta}\right], \quad (3.1c)$$

$$-\frac{1}{\beta}u_\eta + u + v_\theta = 0. \quad (3.1d)$$

A small disturbance is now introduced to the basic flow inside the Stokes layer. Thus the total flow becomes

$$(u, v, w, p) = (u_{\mathrm{B}}, v_{\mathrm{B}}, w_{\mathrm{B}}, p_{\mathrm{B}}) + \epsilon_1(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}),$$

where ϵ_1 is a vanishing small parameter. Substitution into (3.1 *a–d*) and linearization with respect to ϵ_1 yields the following system that governs the evolution of disturbances inside the Stokes layer:

$$\tilde{u}_\tau + \beta T\left(-\frac{1}{\beta}u_{\mathrm{B}\eta}\tilde{u} - \frac{1}{\beta}u_{\mathrm{B}}\tilde{u}_\eta + u_{\mathrm{B}\theta}\tilde{v} + v_{\mathrm{B}}\tilde{u}_\theta - 2v_{\mathrm{B}}\tilde{v}\right)$$
$$-2w_{\mathrm{B}}\tilde{w}\cos\theta = \tilde{p}_\eta - \tfrac{1}{2}\beta^2\frac{\partial}{\partial\theta}\left(-\frac{1}{\beta}\tilde{v}_\eta + \tilde{v} - \tilde{u}_\theta\right), \quad (3.2a)$$

$$v_\tau + \beta T\left(-\frac{1}{\beta}v_{\mathrm{B}\eta}\tilde{u} - \frac{1}{\beta}u_{\mathrm{B}}\tilde{v}_\eta + v_{\mathrm{B}\theta}\tilde{v} + v_{\mathrm{B}}\tilde{v}_\theta + u_{\mathrm{B}}\tilde{v} + v_{\mathrm{B}}\tilde{u}\right)$$
$$+2w_{\mathrm{B}}\tilde{w}\sin\theta = -\beta\tilde{p}_\theta - \tfrac{1}{2}\beta\frac{\partial}{\partial\eta}\left(-\frac{1}{\beta}\tilde{v}_\eta + \tilde{v} - \tilde{u}_\theta\right), \quad (3.2b)$$

$$\tilde{w}_\tau + \beta T\left(-\frac{1}{\beta}w_{\mathrm{B}\eta}\tilde{u} - \frac{1}{\beta}u_{\mathrm{B}}\tilde{w}_\eta + w_{\mathrm{B}\theta}\tilde{v} + v_{\mathrm{B}}\tilde{w}_\theta\right) = \tfrac{1}{2}\beta^2\left(\frac{1}{\beta}\tilde{w}_{\eta\eta} - \frac{1}{\beta}\tilde{w}_\eta + \tilde{w}_{\theta\theta}\right), \quad (3.2c)$$

$$-\frac{1}{\beta}\tilde{u}_\eta + \tilde{u} + \tilde{v}_\theta = 0. \quad (3.2d)$$

The boundary conditions appropriate to the disturbance are those of no slip at the wall, $\eta = 0$, and that the disturbance vanishes as $\eta \to \infty$. The expression $\epsilon^2 = \beta T$ has also been used.

It can be seen from (3.2*a–d*) above, that terms involving θ-derivatives are smaller ($O(\beta)$) than those involving τ and η derivatives and so the θ variations are comparatively slow. Thus a WKBJ multiple scales approach can take account of this, and we consider disturbances of this form next.

Solutions are sought, of the form

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}) = b_1(\theta)(\tilde{u}_1, \tilde{v}_1, \tilde{w}_1, \tilde{p}_1) E + \beta b_2(\theta)(\tilde{u}_2, \tilde{v}_2, \tilde{w}_2, \tilde{p}_2) + O(\beta^2) + \mathrm{C.C.}, \tag{3.3}$$

where c.c. denotes complex conjugate and the perturbed quantities $\tilde{u}_1$, $\tilde{v}_1$, $\tilde{w}_1$, $\tilde{p}_1$, etc. are functions of θ, η, τ. Also, E is given by

$$E = \exp\left(\mathrm{i}\beta^{-1} \int^{\theta} k(\theta')\,\mathrm{d}\theta'\right).$$

In the expression for E the lower limit of integration can be freely chosen; the form of the exponent of the exponential implies that we are looking at disturbances with wavenumbers in the θ-direction corresponding to wavelengths of order β, i.e. the wavelengths are of the order of the Stokes-layer thickness. Such a regime is physically realistic, since it implies that the vortices have roughly square cross-section and are embedded inside the Stokes layer.

The Taylor number T expands in powers of β as follows,

$$T = T_0 + \beta T_1 + \beta^2 T_2 + \ldots, \tag{3.4}$$

and T_0 is taken to be the critical Taylor number of neutral stability.

It should be noted that in the stability problem just posed, disturbances which are independent of ϕ are considered, which means that the vortex amplitudes do not vary as we move down the pipe. Previous investigators (see Seminara & Hall 1976) have found disturbances of this form to provide the most unstable linear modes. There is however the possibility of the local inflexional instabilities found by Tromans (1979), and Cowley (1986) by quasi-steady analyses (the Stokes-layer Reynolds number for our problem is $W(\nu/\omega)^{\frac{1}{2}}$ which is proportional to $(\delta\beta)^{-\frac{1}{2}} \gg 1$). As these modes will not necessarily be observed in practice and owing to evidence that centrifugal Stokes layers have instability characteristics which are well described by Floquet theory (see introduction, and Cowley 1986) we choose to concentrate on the latter form of instability.

A feature that emerges due to the unsteadiness of the basic flow (see also Hall 1984) is that the problem does not exhibit the classical Görtler type of instability, where the flow is stable or unstable depending on whether the local geometry is convex or concave respectively. On the contrary, as the analysis in the following section shows, instability is found on both the inner and outer bends. Although this might seem a little surprising at first, it is in fact consistent with the inviscid limit of the flow. When the inviscid limit is taken, the Taylor number $T \to \infty$ and it can be seen from the definition of T that the Reynolds number for the flow in the θ-direction (based on Stokes-layer thickness) also becomes infinite (in fact $T \sim \bar{v}\beta/\nu$ where $\bar{v}$ is typical velocity in the θ-direction). The timescale becomes very short ($\sim T^{-\frac{1}{2}}$) which implies that the problem reduces to a quasi-steady Taylor problem.

Expansions can be set up in powers of $T^{-\frac{1}{2}}$ and substituted into the governing equations to obtain the inviscid-limit behaviour of the disturbances. We do not present this analysis here, but instead provide a physical argument for the appearance of instability on both inner and outer bends. It can be seen that as $T \to \infty$ the main component of the flow is down the pipe. If Rayleigh's centrifugal instability criterion (cf. Drazin & Reid 1981) is applied to this flow, it is found that there is instability in the inviscid limit at both the outer and inner bends.

4. Stability problem in the limit $\beta \to 0$

Substitution of the expressions (3.3), (3.4) into the governing perturbation equations (3.2*a–d*) defines the stability equations for the disturbance quantities at successive powers of β. It is convenient to introduce the vector

$$\boldsymbol{q}_i = \left[\tilde{p}_i, \frac{\partial \tilde{v}_i}{\partial \eta}, \frac{\partial \tilde{w}_i}{\partial \eta}, \tilde{u}_i, \tilde{v}_i, \tilde{w}_i\right]^{\mathrm{T}} \quad (i = 1, 2, \ldots),$$

to represent perturbation quantities at successive orders in β. Here and in the rest of the paper a superscript T denotes the transpose of a vector or matrix. The governing equations at different orders in β can therefore be written in vector form (as in Eagles 1971 for example) and the two leading-order problems are

$$\boldsymbol{I}\frac{\partial \boldsymbol{q}_1}{\partial \eta} - \boldsymbol{A}\boldsymbol{q}_1 - \boldsymbol{B}\frac{\partial \boldsymbol{q}_1}{\partial \tau} = 0, \tag{4.1}$$

$$b_2(\theta)\left(\boldsymbol{I}\frac{\partial \boldsymbol{q}_2}{\partial \eta} - \boldsymbol{A}\boldsymbol{q}_2 + \boldsymbol{B}\frac{\partial \boldsymbol{q}_2}{\partial \tau}\right) = \boldsymbol{L}\boldsymbol{q}_1. \tag{4.2}$$

In (4.1), (4.2) above, $\boldsymbol{I}$ is the identity matrix of size 6×6 and $\boldsymbol{A}, \boldsymbol{B}$ are the 6×6 matrices given by

$$\boldsymbol{A} = \begin{bmatrix} 0 & -\tfrac{1}{2}\mathrm{i}k & 0 & \tfrac{1}{2}k^2 + \mathrm{i}kT_0 v_{\mathrm{B0}} & 0 & -2w_{\mathrm{B0}}\cos\theta \\ 2\mathrm{i}k & 0 & 0 & -2T_0 v_{\mathrm{B0}\eta} & k^2 + 2\mathrm{i}kT_0 v_{\mathrm{B0}} & 4w_{\mathrm{B0}}\sin\theta \\ 0 & 0 & 0 & -T_0 w_{\mathrm{B0}\eta} & 0 & k^2 + 2\mathrm{i}kT_0 v_{\mathrm{B0}} \\ 0 & 0 & 0 & 0 & \mathrm{i}k & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{4.3}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

The matrix $\boldsymbol{L}$ that appears on the right-hand side of (4.2) is a 6×6 matrix with elements which contain $b_1(\theta)$, $\mathrm{d}b_1/\mathrm{d}\theta$ as well as $O(\beta)$ basic flow quantities. These elements are given in the Appendix.

A differential equation that governs the behaviour of the amplitude $b_1(\theta)$ can now be obtained by imposition of a solvability condition on equation (4.2). By this we mean that if equation (4.2) is to have non-trivial solutions, then its right-hand side must be orthogonal to the adjoint solution of the homogeneous equivalent of (4.2); this equation is the same as (4.1), thus it is sufficient to find adjoint solutions to (4.1).

If the governing equations (4.1), (4.2) are considered in operator form by the introduction of the operator $\mathbf{C}$ given by

$$\mathbf{C} \equiv \boldsymbol{I}\frac{\partial}{\partial \eta} - \boldsymbol{A} + \boldsymbol{B}\frac{\partial}{\partial \tau}, \tag{4.5}$$

and if V is the adjoint solution of (4.1), then by virtue of $\mathbf{C}(\boldsymbol{q}_1)$ being zero we obtain

$$V^{\mathrm{T}}\,\mathbf{C}(\boldsymbol{q}_1) = 0. \tag{4.6}$$

Equation (4.6) can now be used to obtain the adjoint equation, together with its solution, as follows. We integrate (4.6) with respect to η and τ; the range of η is $[0, \infty]$ (the whole extent of the Stokes layer) and that of τ is $[0, 2\pi]$ since the basic flow is periodic with period 2π. Thus we obtain

$$\int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} V^{\mathrm{T}}\left(\boldsymbol{I}\frac{\partial \boldsymbol{q}_1}{\partial \eta} - \boldsymbol{A}\boldsymbol{q}_1 + \boldsymbol{B}\frac{\partial \boldsymbol{q}_1}{\partial \tau}\right)\mathrm{d}\tau\,\mathrm{d}\eta = 0.$$

After change of the order of integration, and integration by parts, we have

$$\int_0^{2\pi}[V^{\mathrm{T}}\boldsymbol{I}\boldsymbol{q}_1]_{\eta=0}^{\infty}\,\mathrm{d}\tau + \int_{\eta=0}^{\infty}[V^{\mathrm{T}}\boldsymbol{B}\boldsymbol{q}_1]_{\tau=0}^{2\pi}\,\mathrm{d}\eta - \int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi}\left[\frac{\partial V^{\mathrm{T}}}{\partial \eta}\boldsymbol{I}\boldsymbol{q}_1 + V^{\mathrm{T}}\boldsymbol{A}\boldsymbol{q}_1 + \frac{\partial V^{\mathrm{T}}}{\partial \tau}\boldsymbol{B}\boldsymbol{q}_1\right]\mathrm{d}\tau\,\mathrm{d}\eta = 0. \tag{4.7}$$

Equation (4.7), above, can be satisfied by choosing the components of V ($V_i = 1.6$ say) as follows. V_1, V_2, V_3 are taken to vanish at $\eta = 0$, $\eta = \infty$ and V is taken to be periodic in τ with period 2π, together with the adjoint equation

$$\mathbf{C}^{\dagger}(V) = 0,$$

where

$$\mathbf{C}^{\dagger} = \boldsymbol{I}\frac{\partial}{\partial \eta} + \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}\frac{\partial}{\partial \tau}. \tag{4.8}$$

The periodicity condition on V is valid as we are only considering neutral solutions (cf. (3.4)). Thus the orthogonality condition on (4.2) described earlier yields

$$\int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} V^{\mathrm{T}}\boldsymbol{L}\boldsymbol{q}_1\,\mathrm{d}\tau\,\mathrm{d}\eta = 0, \tag{4.9}$$

where V satisfies (4.8). This condition now leads to the following ordinary differential equation for $b_1(\theta)$:

$$J(\theta)\frac{\mathrm{d}b_1}{\mathrm{d}\theta} + H(\theta)\,b_1 = 0. \tag{4.10}$$

The functions $J(\theta)$, $H(\theta)$ are double integrals over η- and τ-space and are given in the Appendix.

Thus far, we have identified the evolution equations for the leading-order and $O(\beta)$ perturbations and obtained a differential equation that describes the behaviour of the leading-order vortex amplitude. The calculations are performed at the critical value of the Taylor number T_0 (fixed for some θ) and wavenumber $k(\theta)$. It should be noted that $k(\theta)$ is complex for general θ and due to its θ-dependence we would expect the vortices at some positions to have larger amplitudes than those in other positions. The complex k effect is expected to provide the decay of the vortex amplitude with θ (it is in connection with these eigenvalues that the problem has to be resolved mathematically in the complex θ-plane; this is explained fully in the next section). Our aim is to identify where the largest vortex amplitudes first occur (in particular if they do so at the inner or outer bend of the pipe) and to calculate the critical Taylor number above which the flow becomes unstable.

Suppose, for definiteness, that instability first sets in at the outer bend $\theta = 0$ (such positions are in fact found to be more unstable by our numerical calculations described in §§6 and 7). It is important, therefore, to analyse the behaviour of b_1 near $\theta = 0$. In order to fix the neutral k and T_0 at $\theta = 0$, it is convenient to expand these quantities in power series in θ (for $\theta \ll 1$) and consider leading-order quantities. Thus, for $\theta \ll 1$, we write

$$k = k_0 + \theta k_1 + \theta^2 k_2 + \dots, \tag{4.11}$$

$$T_0 = \overline{T}_0 + \theta^2 \overline{T}_1 + O(\theta^4). \tag{4.12}$$

The form of the expression for T_0 follows from the fact that near a critical point of the marginal stability curve, $(\overline{T}_0, k_0)$ say, T_0 has a minimum as a function of k and so $T_0 - \overline{T}_0 \sim (k-k_0)^2$. The matrix $\boldsymbol{A}$ and the vector $\boldsymbol{q}_1$ expand as follows:

$$\boldsymbol{A} = \overline{\boldsymbol{A}}_0 + \theta \overline{\boldsymbol{A}}_1 + \theta^2 \overline{\boldsymbol{A}}_2 + \dots, \tag{4.13}$$

$$\boldsymbol{q}_1 = \boldsymbol{q}_{10} + \theta \boldsymbol{q}_{11} + \theta^2 \boldsymbol{q}_{12} + \dots . \tag{4.14}$$

If we now substitute (4.11)–(4.14) into (4.1) and equate coefficients of successive powers of θ, we obtain

$$O(\theta): \quad \mathbf{C}_0(\boldsymbol{q}_{10}) = 0, \tag{4.15}$$

$$O(\theta^2): \quad \mathbf{C}_0(\boldsymbol{q}_{11}) = \overline{\boldsymbol{A}}_1 \boldsymbol{q}_{10}, \tag{4.16}$$

where $\mathbf{C}_0$ denotes the operator $\mathbf{C}$ evaluated at $\theta = 0$. Thus for non-trivial solutions of (4.16) to exist we must satisfy the following orthogonality condition,

$$\int_{\eta=0}^{\infty} \int_{\tau=0}^{2\pi} \boldsymbol{V}_0^{\mathrm{T}} \overline{\boldsymbol{A}}_1 \boldsymbol{q}_{10} \, \mathrm{d}\tau \, \mathrm{d}\eta = 0, \tag{4.17}$$

where $\mathbf{C}_0^{\dagger}(\boldsymbol{V}_0) = 0$. If we now write $\boldsymbol{V}_0 = (V_{01}, V_{02}, V_{03}, V_{04}, V_{05}, V_{06})^{\mathrm{T}}$, it follows from (4.17) that

$$k_1 = M/N,$$

where

$$M = \int_{\eta=0}^{\infty} \int_{\tau=0}^{2\pi} (\mathrm{i}k_0 \overline{T}_0 f_\eta \, V_{01} \tilde{u}_{10} - 2\overline{T}_0 f_{\eta\eta} \, V_{02} \tilde{u}_{10} + 2\mathrm{i}k_0 \overline{T}_0 f_\eta \, V_{02} \tilde{v}_{10}$$
$$+ 4w_{\mathrm{B}0} \, V_{02} \tilde{w}_{10} + 2\mathrm{i}k_0 \overline{T}_0 f_\eta \, V_{03} \tilde{w}_{10}) \, \mathrm{d}\tau \, \mathrm{d}\eta, \tag{4.18a}$$

$$N = \int_{\eta=0}^{\infty} \int_{\tau=0}^{2\pi} [V_{01}(\tfrac{1}{2}\mathrm{i}\tilde{v}_{10\eta} - k_0 \tilde{u}_{10}) - V_{02}(2\mathrm{i}\tilde{p}_{10} + 2k_0 \tilde{v}_{10})$$
$$- 2k_0 \, V_{03} \tilde{w}_{10} - \mathrm{i} V_{04} \tilde{v}_{10}] \, \mathrm{d}\tau \, \mathrm{d}\eta. \tag{4.18b}$$

Now the condition that $\overline{T}_0$ is a minimum as a function of k_0, with k_0 real, may be obtained by differentiating (4.15) with respect to k_0 and setting $\partial \overline{T}_0 / \partial k_0 = 0$. This gives

$$\mathbf{C}_0 \left(\frac{\partial \boldsymbol{q}_{10}}{\partial k_0} \right) = \frac{\partial \overline{\boldsymbol{A}}_0}{\partial k_0} \boldsymbol{q}_{10}. \tag{4.19a}$$

Hence for non-trivial solutions of (4.19*a*) we require

$$\int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} V_0^{\mathrm{T}}\frac{\partial \bar{\boldsymbol{A}}_0}{\partial k_0}\boldsymbol{q}_{10}\,\mathrm{d}\tau\,\mathrm{d}\eta = 0. \tag{4.19b}$$

When the condition (4.19*b*) is written out in full, it is seen to be identical to the expression (4.18*b*) for N. Thus we conclude that $N = 0$, and hence the expansions of k and $\boldsymbol{q}$ in powers of θ about $k_0, \boldsymbol{q}_{10}$ are not regular. Such behaviour was also found by Walton (1978) in his investigation of the stability of the steady flow in a narrow spherical annulus. An alternative expansion that does not produce singular behaviour in k_1 is found to proceed in powers of $\theta^{\frac{1}{2}}$. We write

$$\left.\begin{aligned}
k &= k_0+\theta^{\frac{1}{2}}\tilde{k}_1+\theta\tilde{k}_2+O(\theta^{\frac{3}{2}}),\\
\boldsymbol{q}_1 &= \boldsymbol{q}_{10}+\theta^{\frac{1}{2}}\tilde{k}_1\boldsymbol{q}_{11}+\theta(\boldsymbol{q}_{12}+\tilde{k}_2\boldsymbol{q}_{11})+O(\theta^{\frac{3}{2}}),\\
\boldsymbol{A} &= \bar{\boldsymbol{A}}_0+\theta^{\frac{1}{2}}\tilde{\boldsymbol{A}}_1+\theta\tilde{\boldsymbol{A}}_2+O(\theta^{\frac{3}{2}}),\\
T_0 &= \bar{T}_0+\theta\bar{T}_1+O(\theta^2),\\
V &= V_0+\theta^{\frac{1}{2}}\tilde{k}_1V_1+\theta(V_2+\tilde{k}_2V_1)+O(\theta^{\frac{3}{2}}).
\end{aligned}\right\} \tag{4.20}$$

It should be noted here that although the quantities k_0, $\boldsymbol{q}_{10}$, $\bar{\boldsymbol{A}}_0$, $\bar{T}_0$ and V_0 are the same as those occurring in the expansions (4.10)–(4.13), the terms at $O(\theta^{\frac{1}{2}})$ and higher are of course not equivalent to the $O(\theta)$ and higher terms in (4.11)–(4.14), even though the same notation has been used.

Substitution of the expansions (4.20) into (4.1) yields

$$O(\theta^0):\quad \mathbf{C}_0(\boldsymbol{q}_{10}) = 0, \tag{4.21a}$$

$$O(\theta^{\frac{1}{2}}):\quad \tilde{k}_1\mathbf{C}_0(\boldsymbol{q}_{11}) = \tilde{\boldsymbol{A}}_1\boldsymbol{q}_{10}, \tag{4.21b}$$

$$O(\theta):\quad \mathbf{C}_0(\boldsymbol{q}_{12}) = -\frac{\tilde{k}_2}{\tilde{k}_1}\mathbf{C}_0(\boldsymbol{q}_{11})+\tilde{k}_1\tilde{\boldsymbol{A}}_1\boldsymbol{q}_{11}+\tilde{\boldsymbol{A}}_2\boldsymbol{q}_{10}. \tag{4.21c}$$

The condition that (4.21*c*) possesses non-trivial solutions gives an equation for $\tilde{k}_1^2$ in terms of k_0. This is

$$\tilde{k}_1^2 = -M_1/N_1,$$

where

$$\begin{aligned}
M_1 = \int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi}[&V_{01}(\mathrm{i}k\bar{T}_0f_\eta\tilde{u}_{10})+V_{02}(-2\bar{T}_0f_{\eta\eta}\tilde{u}_{10}+2\mathrm{i}k_0\bar{T}_0f_\eta\tilde{v}_{10}+4w_{\mathrm{B}0}\tilde{w}_{10})\\
&+V_{03}(-2w_{\mathrm{B}0\eta}\bar{T}_1\tilde{u}_{10}+2\mathrm{i}k_0\bar{T}_0f_\eta\tilde{w}_{10})]\,\mathrm{d}\tau\,\mathrm{d}\eta,
\end{aligned} \tag{4.22a}$$

$$\begin{aligned}
N_1 = \int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi}[&V_{01}(\tfrac{1}{2}\tilde{u}_{10}-\tfrac{1}{2}\mathrm{i}\tilde{v}_{11\eta}+k_0\tilde{u}_{11})+V_{02}(\tilde{v}_{10}+2\mathrm{i}\tilde{p}_{11}+2k_0\tilde{v}_{11})\\
&+V_{03}(\tilde{w}_{10}+2k_0\tilde{w}_{11})+\mathrm{i}V_{04}\tilde{v}_{11}]\,\mathrm{d}\tau\,\mathrm{d}\eta.
\end{aligned} \tag{4.22b}$$

The behaviour of b_1 as $\theta\to 0$ can now be examined. To do this the coefficients $J(\theta)$, $H(\theta)$ of the amplitude equation (4.10) are expanded in powers of $\theta^{\frac{1}{2}}$. These expansions are (see Appendix 1),

$$J(\theta) = J_0+\theta^{\frac{1}{2}}J_1+O(\theta), \tag{4.23a}$$

$$H(\theta) = \theta^{-\frac{1}{2}}H_0+O(1). \tag{4.23b}$$

It is found that J_0 is equivalent to the expression (4.18*b*) for N and by (4.19*b*) is therefore zero. Next we obtain

$$\begin{aligned} J_1 = \int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} [\tilde{k}_1 V_{01}(-\tfrac{1}{2}\tilde{v}_{1\eta\eta} - \mathrm{i}k_0\tilde{u}_{11} - \mathrm{i}\tilde{u}_{10}) + 2\mathrm{i}\tilde{k}_1 V_{02}(\tilde{p}_{11} - \mathrm{i}k_0\tilde{v}_{11} - \mathrm{i}\tilde{v}_{10}) \\ -2\mathrm{i}\tilde{k}_1 V_{03}(k_0\tilde{w}_{11} + \tilde{w}_{10}) + \tilde{k}_1 V_{04}\tilde{v}_{11} - \tilde{k}_1 V_{11}(\tfrac{1}{2}\tilde{v}_{10\eta} + k_0\tilde{u}_{10}) \\ +2\tilde{k}_1 V_{12}(\tilde{p}_{10} - \mathrm{i}k_0\tilde{v}_{10}) - 2\mathrm{i}k_0 V_{13}\tilde{w}_{10}\tilde{k}_1 + \tilde{k}_1 V_{14}\tilde{v}_{10}]\,\mathrm{d}\tau\,\mathrm{d}\eta, \end{aligned} \tag{4.24a}$$

$$\begin{aligned} H_0 = \int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} [\tilde{k}_1 V_{01}(-\tfrac{1}{4}\tilde{v}_{11\eta} - \tfrac{1}{4}\mathrm{i}\tilde{u}_{10} - \tfrac{1}{2}\mathrm{i}k_0\tilde{u}_{11}) + \tilde{k}_1 V_{02}(\tilde{p}_{11} - \tfrac{1}{2}\mathrm{i}\tilde{v}_{10} - \mathrm{i}k_0\tilde{v}_{11}) \\ -\mathrm{i}\tilde{k}_1 V_{03}(\tfrac{1}{2}\tilde{w}_{10} + k_0\tilde{w}_{11}) + \tfrac{1}{2}\tilde{k}_1 V_{04}\tilde{v}_{11}]\,\mathrm{d}\tau\,\mathrm{d}\eta. \end{aligned} \tag{4.24b}$$

We can now show that H_0 is a multiple of J_1. To obtain this linear relation, we consider the adjoint equation (4.7). If we use the expansions (4.20) near $\theta = 0$, we obtain the following equations at successive powers of $\theta^{\frac{1}{2}}$.

$$O(\theta^0):\quad \mathbf{C}_0^\dagger(\boldsymbol{V}_0) = 0, \tag{4.25a}$$

$$O(\theta^{\frac{1}{2}}):\quad \tilde{k}_1\mathbf{C}_0^\dagger(\boldsymbol{V}_1) = -\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{V}_0, \tag{4.25b}$$

$$O(\theta):\quad \mathbf{C}_0^\dagger(\boldsymbol{V}_2) = \frac{\tilde{k}_2}{\tilde{k}_1}\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{V}_0 - \tilde{k}_1\tilde{\boldsymbol{A}}_1^{\mathrm{T}}\boldsymbol{V}_1 + \tilde{\boldsymbol{A}}_2^{\mathrm{T}}\boldsymbol{V}_0. \tag{4.25c}$$

For non-trivial solutions of (4.25*c*) the following condition must hold:

$$\begin{aligned} \tilde{k}_1^2\int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} [V_{11}(-\tfrac{1}{2}\mathrm{i}\tilde{v}_{10\eta} + k_0\tilde{u}_{10}) + 2V_{12}(\mathrm{i}\tilde{p}_{10} + k_0\tilde{v}_{10}) + 2k_0 V_{13}\tilde{w}_{10} + \mathrm{i}V_{14}\tilde{v}_{10}]\,\mathrm{d}\tau\,\mathrm{d}\eta \\ = -\int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} [\mathrm{i}k_0\overline{T}_0 f_\eta \mathbf{u}_{10} V_{01} + 2V_{02}(-\overline{T}_0 f_{\eta\eta}\tilde{u}_{10} + \mathrm{i}k_0\overline{T}_0 f_\eta\tilde{v}_{10} + 2w_{\mathrm{B}0}\tilde{w}_{10}) \\ +2V_{03}(-w_{\mathrm{B}0\eta}\tilde{T}_1\tilde{u}_{10} + \mathrm{i}k_0\overline{T}_0 f_\eta\tilde{w}_{10})]\,\mathrm{d}\tau\,\mathrm{d}\eta \\ -\tilde{k}_1^2\int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi}(\tfrac{1}{2}V_{01}\tilde{u}_{10} + V_{02}\tilde{v}_{10} + V_{03}\tilde{w}_{10})\,\mathrm{d}\tau\,\mathrm{d}\eta. \end{aligned} \tag{4.26}$$

Substitution for $\tilde{k}_1^2$ from (4.22*a*, *b*) into (4.26) yields

$$H_0 = \tfrac{1}{4}J_1.$$

Thus as $\theta \to 0$ the amplitude equation (4.10) can be written as

$$2(\theta^{\frac{1}{2}} + \gamma_1\theta + \ldots)\frac{\mathrm{d}b_1}{\mathrm{d}\theta} + \tfrac{1}{2}(\theta^{-\frac{1}{2}} + \gamma_0 + \ldots)\,b_1 = 0, \tag{4.27}$$

where γ_0, γ_1 are constants given by double integrals of the expanded quantities. As $\theta \to 0$, therefore, (4.27) has solutions which behave like

$$b_1(\theta) \sim \Gamma_1\theta^{-\frac{1}{4}}\exp((\gamma_1 - \gamma_0)\,\theta^{\frac{1}{2}}), \tag{4.28}$$

where Γ_1 is a constant.

Equation (4.28) shows that b_1 becomes singular as $\theta \to 0$. A natural assumption would be that this singularity can be smoothed out by a rescaling of θ and a formation of an inner expansion near $\theta = 0$. It turns out, however, that such a procedure leads to unacceptable results which can only be resolved by the analytic continuation of the solution to the complex θ-plane. The need for such a procedure, together with the solution that overcomes singular behaviour on the real θ-axis, are given in the following section.

It should be noted here that the analysis just carried out is also valid in the neighbourhood of $\theta = \pi$ (inner bend), by a slight modification of the elements of the leading-order matrix $\bar{\boldsymbol{A}}_0$.

5. The inner solution

The remarks made at the end of the last section on the need for the solution to be considered in the complex θ-plane can now be clarified. First of all it must be noted that this is required because of the properties of the asymptotic solutions, which imply that a scaling of θ on the real axis does not achieve the correct results. As can be seen from the expansions (3.3), and more specifically the exponent of the exponential near $\theta = 0$, the leading scales to be reconsidered on the real axis are $\theta = O(\beta^{\frac{2}{3}})$ and $\theta = O(\beta)$. This procedure was adopted by Walton (1978) in the narrow spherical annulus problem, after the assumption that the critical Taylor number at the equator is the same as that for the corresponding Taylor-cylinder problem. The equation that determines the evolution of the leading-order vortex amplitude is then found to be an Airy equation, which has solutions that decay at infinity but exhibit oscillatory behaviour at minus infinity. Such solutions are unacceptable from a physical point of view; the problem was resolved by Soward & Jones in 1983. We now describe briefly the development of their method applied to our problem.

The main aim in problems of this kind is to obtain a dispersion relation for the Taylor number which can then be calculated by fixing some quantities while varying others. This dispersion relation takes the form

$$T = T(k, \theta, \sigma) + O(\beta), \tag{5.1}$$

where σ is a growth rate (in our case obtained by the Fourier analysis of solutions in time) and k, θ are as defined earlier. The Taylor number T can now be regarded as a function of the three complex variables k, θ, σ but it must of course remain real and constant for all θ in a physical flow (e.g. experiments where it is fixed). It can be shown that at a minimum of T the following conditions must hold,

$$\mathrm{Re}\left(\frac{T_k}{T_\sigma}\right) = \mathrm{Re}\left(\frac{T_\theta}{T_\sigma}\right) = 0.$$

The above conditions are satisfied if

$$T_k = T_\theta = 0. \tag{5.2}$$

The necessity of satisfying the conditions (5.2) is crucial in the analysis if physically acceptable solutions are to be found. In a lot of problems these conditions are satisfied on the real axis of θ (as in Hall 1984, for example), but it can be seen that complex values of θ are also permissible. Mathematically, if $T_k = T_\theta = 0$, the amplitude equation for linear perturbations turns out to be a parabolic cylinder equation (see Hall 1984), which has solutions that decay exponentially as $|\theta| \to \infty$. If $T_\theta \neq 0$, however, as is the case for the narrow spherical annulus Taylor problem studied by Walton, and Soward & Jones, the resulting amplitude equation is an Airy-type equation which does not provide the correct solutions. It should be noted that both the amplitude equations mentioned above arise after a normal mode analysis in time, or in the case of a time-dependent basic flow (as in Hall 1984 and the present paper) by a Fourier expansion in time. The mathematical details, and scalings involved on the coordinates that governs the vortex strength, can be found in Soward & Jones (1983).

The important feature that follows from problems like our curved-pipe flow or the flow in a narrow spherical annulus, is that acceptable solutions that describe the vortex amplitudes at the critical Taylor number (or, conversely, the correct solution of the eigenvalue problem that determines the critical Taylor number above which instability occurs), can only be found if solutions are considered at the saddle point of T, where $T_k = T_\theta = 0$ (to use our notation). As the saddle point is not on the real axis we must search for it by continuing the solution into the complex plane of θ. When such a point is identified, the solution in its neighbourhood is not just an inner solution that smooths out the singularity found on the real axis, but provides a set of asymptotic solutions that match with ones valid away from the saddle point and which describe the flow (in the limit $\beta \rightarrow 0$) for all real values of θ as well. The eigenvalue problem, together with the search for the saddle point, and consequently the identification of the correct value of T, must be solved numerically. Before this is done we describe the solutions in the neighbourhood of the saddle point ('inner solutions') and show that they form rational asymptotic solutions in the limit $\beta \rightarrow 0$. This is desirable if we are to illustrate that the remarks made earlier are consistent with the analysis.

Suppose, therefore, that the position of the saddle point is at $\theta = \theta_0$. We wish to study symmetric modes, in this case modes symmetric with respect to θ. It follows that

$$\mathrm{Re}\,(\theta_0) = \mathrm{Im}\,(k_0) = 0. \tag{5.3}$$

The first relationship in (5.3) above follows from the symmetry of the problem. With this in mind, it can be seen that the leading-order governing equations (see equations (6.2a,b) of §6 which follow from (4.1)) have real coefficients if $\mathrm{Im}\,(k_0) = 0$.

If we consider asymptotic solutions to the problem of the WKBJ type (cf. (3.3) also), we can write down two linearly independent ones of the form

$$\boldsymbol{q}_\pm = d_\pm(\theta) \exp\{\pm \mathrm{i}\beta^{-1} \int^\theta k(\theta')\,\mathrm{d}\theta'\}\, \boldsymbol{q}_{1\pm}(\theta, \eta, \tau) + O(\beta). \tag{5.4}$$

These solutions provide good approximations to the exact solutions when the limit $\beta \rightarrow 0$ is taken, provided that θ is not near a transition point where $k(\theta)$ vanishes. As there are no transition points of $k(\theta)$ on the real axis, it might be assumed that $\boldsymbol{q}_+$ provides an asymptotic solution for the problem. The error associated with this solution as compared to the true one is $O(\beta)$ times the asymptotic solution (i.e. $O(\beta \boldsymbol{q}_+)$). As θ varies in the complex plane, $\boldsymbol{q}_+$ is a valid solution provided that the error associated with it remains $O(\beta)$ times $\boldsymbol{q}_+$. It is found, however, that as θ varies, a domain is encountered where the error is no longer $O(\beta \boldsymbol{q}_+)$ but changes to $O(\beta \boldsymbol{q}_-)$. In this region the error now becomes exponentially large compared to the original asymptotic solution $\boldsymbol{q}_+$ and so $\boldsymbol{q}_+$ is no longer a valid solution here. The boundaries across which such a change takes place are the anti-Stokes lines; the reason that they are present in the description of solutions that are strictly continuous is due to the asymptotic representation and the errors associated with that. For a full description of phase-integral methods refer to Heading (1962).

In order to resolve the difficulty and identify the correct asymptotic solutions, the transition points from which the anti-Stokes lines emanate must be found. The

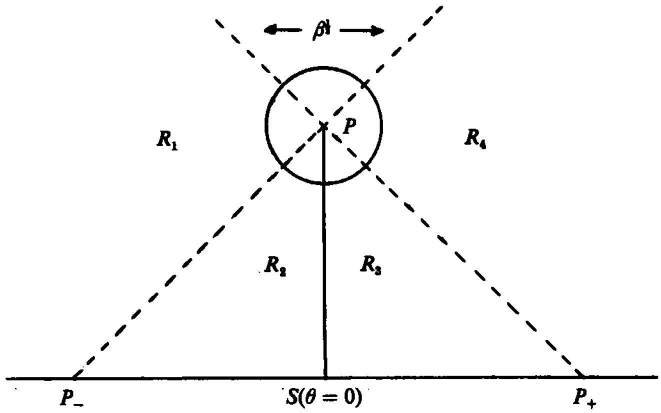

FIGURE 2. An illustration of the various domains into which the complex θ-plane is divided. The inner solution has a domain of validity inside the circle of radius $O(\beta^{\frac{1}{3}})$, and the dominance or subdominance of the outer WKBJ solutions depends on the regions R_1, R_2, R_3, R_4.

problem posed by (4.1)–(4.4) is an eigenvalue problem which has a solution only when an eigenrelation of the following form is satisfied:

$$F(k, \theta, T) = 0. \tag{5.5}$$

The dependence of F on k, θ, T follows from the search of neutrally stable solutions (growth rate $\sigma = 0$) and by a Fourier analysis of solutions in time. This is clarified further in the next section.

It can be seen from the conditions (5.2) that $T_k = 0$ implies a repeated root of (5.5) at $\theta = \theta_0$. This root is taken to be

$$k^{(1)}(\theta_0) = k^{(2)}(\theta_0) = k_0. \tag{5.6}$$

Hence, the point $P(\theta = \theta_0)$ in the complex θ-plane defines a transition point in whose vicinity the WKBJ solutions (3.3) become identical. The other condition, $T_\theta = 0$, implies that in the limit $\beta \to 0$ the transition points coincide at $\theta = \theta_0$. This property indicates that there exist anti-Stokes lines that divide the complex θ-plane into four regions. They are defined by

$$\mathrm{Im}\left\{\int_{\theta_0}^{\theta} (k^{(2)} - k^{(1)})\, \mathrm{d}\theta'\right\} = 0. \tag{5.7a}$$

There are also distinct Stokes lines defined by

$$\mathrm{Re}\left\{\int_{\theta_0}^{\theta} (k^{(2)} - k^{(1)})\, \mathrm{d}\theta'\right\} = 0. \tag{5.7b}$$

We have not calculated the exact shape of the Stokes and anti-Stokes lines from (5.7a, b), but in figure 2 we present a schematic of how the complex θ-plane is divided up by the Stokes and anti-Stokes lines (see also Soward & Jones 1983).

Our concern is with symmetric modes, so the Stokes line PS is along the imaginary θ-axis while the anti-Stokes lines PP_-, PP_+ meet the real θ-axis at $P_-(\theta = -\theta_1)$, $P_+(\theta = \theta_1)$ respectively. As explained earlier, if the two solutions q_1, q_2 are considered, corresponding to $k^{(1)}$, $k^{(2)}$ respectively, one solution is expected to be dominant in the regions $R_1 + R_4$ and subdominant in $R_2 + R_3$, while the other solution is dominant in $R_2 + R_3$ and subdominant in $R_1 + R_4$. An asymptotic solution which approximates the true solution uniformly along the real θ-axis can be found by incorporating a

Stokes constant ζ say; the solutions now become (see also Heading 1962; Soward & Jones 1983)

$$\boldsymbol{q} = \boldsymbol{q}_1 \qquad (\theta \leqslant 0), \tag{5.8a}$$

$$\boldsymbol{q} = \boldsymbol{q}_1 - \zeta \boldsymbol{q}_2 \quad (\theta > 0). \tag{5.8b}$$

Other WKBJ solutions that correspond to roots of (5.6) other than $k^{(1)}$, $k^{(2)}$ are not required in the construction of (5.9a, b) above, because these solutions are not related to the transition point $\theta = \theta_0$. Thus, if we are to approximate the solution on the real θ-axis by $\boldsymbol{q} = \boldsymbol{q}_1$ the requirement is that ζ, which is a function of the Taylor number, must be zero. The disturbance then takes its maximum value at the point S shown in figure 2. The value of ζ is determined by obtaining an inner solution for $|\theta - \theta_0| = O(\beta^{\frac{1}{2}})$ and matching with the outer solution. Now, the asymptotic solutions (5.4) are valid for complex values of θ, provided that $|\theta-\theta_0| \gg \beta^{\frac{1}{2}}$, and they govern the matching with the inner solution.

We now present the construction of an inner solution; this serves as a check on the analysis presented so far, as well as being a prerequisite for nonlinear evolution of disturbances. An inner variable $\Theta = O(1)$ is considered, defined by

$$\theta - \theta_0 = \beta^{\frac{1}{2}} \Theta. \tag{5.9}$$

The perturbation vector $\boldsymbol{q}$ is expanded in powers of $\beta^{\frac{1}{2}}$ as follows:

$$\boldsymbol{q} = d_1(\Theta)\, \hat{\boldsymbol{q}}_1 E + \beta^{\frac{1}{2}} d_2(\Theta)\, \hat{\boldsymbol{q}}_2 E + \beta d_3(\Theta)\, \hat{\boldsymbol{q}}_3 E + \ldots + \text{c.c.}, \tag{5.10}$$

where $E = \exp[\mathrm{i}k_0 \Theta / \beta^{\frac{1}{2}}]$. The form of E follows by consideration of the more general solution (3.3) near $\theta = \theta_0$. Substitution of (5.9), (5.10) into the governing equations (3.2a–d) yields at $O(\beta^0)$

$$\boldsymbol{I} \frac{\partial \hat{\boldsymbol{q}}_1}{\partial \eta} - \hat{\boldsymbol{A}}_0 \hat{\boldsymbol{q}}_1 + \boldsymbol{B} \frac{\partial \hat{\boldsymbol{q}}_1}{\partial \tau} = 0, \tag{5.11}$$

where $\hat{\boldsymbol{A}}_0$ is the matrix $\boldsymbol{A}$ (given by (4.3)) evaluated at $\theta = \theta_0$, $k = k_0$ and $\boldsymbol{B}$ is the matrix (4.4). Next, at $O(\beta^{\frac{1}{2}})$ we obtain

$$d_2(\Theta) \left[\boldsymbol{I} \frac{\partial \hat{\boldsymbol{q}}_2}{\partial \eta} - \hat{\boldsymbol{A}}_0\, \hat{\boldsymbol{q}}_2 + \boldsymbol{B} \frac{\partial \hat{\boldsymbol{q}}_2}{\partial \tau} = d_{1\Theta} \boldsymbol{M}_1 \hat{\boldsymbol{q}}_1 + \Theta d_1 \boldsymbol{M}_2 \hat{\boldsymbol{q}}_2 + d_1 \boldsymbol{M}_3 \hat{\boldsymbol{q}}_1 \right]. \tag{5.12}$$

The matrices $\boldsymbol{M}_1, \boldsymbol{M}_2, \boldsymbol{M}_3$ are given by

$$\boldsymbol{M}_1 = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & (T_0 f_\eta \sin\theta_0 - \mathrm{i}k_0) & 0 & 0 \\ 2 & 0 & 0 & 0 & 2(T_0 f_\eta \sin\theta_0 - \mathrm{i}k_0) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(T_0 f_\eta \sin\theta_0 - \mathrm{i}k_0) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.13a}$$

$$\boldsymbol{M}_2 = \begin{bmatrix} 0 & 0 & 0 & \mathrm{i}k_0 T_0 f_\eta \cos\theta_0 & 0 & 2w_{\mathrm{B}0} \sin\theta_0 \\ 0 & 0 & 0 & -2T_0 f_{\eta\eta} \cos\theta_0 & 2\mathrm{i}k_0 T_0 f_\eta \cos\theta_0 & 4w_{\mathrm{B}0} \cos\theta_0 \\ 0 & 0 & 0 & 0 & 0 & 2\mathrm{i}k_0 T_0 f_\eta \cos\theta_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.13b}$$

$$\boldsymbol{M}_3 = \boldsymbol{M}_1 \frac{\partial}{\partial \Theta}. \tag{5.13c}$$

We can now find a solution to (5.12) by writing

$$d_2\hat{q}_2 = -\mathrm{i}d_{1\Theta}\hat{q}_2^{(1)} + \Theta d_1\hat{q}_2^{(2)} + d_1\hat{q}_2^{(3)}, \tag{5.14}$$

where $\hat{q}_2^{(1)}$, $\hat{q}_2^{(2)}$, $\hat{q}_2^{(3)}$ are to be determined. Substituting (5.14) into (5.12) and picking out coefficients of $d_{1\Theta}$, Θd_1 and d_1 yields equations for $\hat{q}_2^{(1)}$, $\hat{q}_2^{(2)}$, $\hat{q}_2^{(3)}$ respectively. If we denote the operator $\hat{\mathbf{C}}_0$ by

$$\hat{\mathbf{C}}_0 = \boldsymbol{I}\frac{\partial}{\partial\eta} - \hat{\boldsymbol{A}}_0 + \boldsymbol{B}\frac{\partial}{\partial\tau}, \tag{5.15a}$$

these equations are

$$\hat{\mathbf{C}}_0(\hat{q}_2^{(1)}) = -\hat{\mathbf{C}}_{k_0 0}(\hat{q}_1), \tag{5.15b}$$

$$\hat{\mathbf{C}}_0(\hat{q}_2^{(2)}) = -\hat{\mathbf{C}}_{\theta_0 0}(\hat{q}_1), \tag{5.15c}$$

$$\hat{\mathbf{C}}_0(\hat{q}_2^{(3)}) = -\hat{\mathbf{C}}_{k_0 0}(\hat{q}_{1\Theta}). \tag{5.15d}$$

In (5.15*b*–*d*) above the subscripts k_0, θ_0 denote partial differentiation with respect to k_0, θ_0 respectively and a subscript zero means evaluation of quantities at $\theta = \theta_0$.

At the next order, $O(\beta)$, the equation for $\hat{q}_3$ is found to be

$$\begin{aligned} d_3(\Theta)\,\hat{\mathbf{C}}_0(\hat{q}_3) = {} & d_{1\Theta\Theta}[\boldsymbol{M}^{(1)}\hat{q}_1 + \boldsymbol{M}^{(2)}\hat{q}_2^{(1)}] \\ & + \Theta d_{1\Theta}[\boldsymbol{N}^{(1)}\hat{q}_1 + \boldsymbol{N}^{(2)}\hat{q}_2^{(1)} + \boldsymbol{N}^{(3)}\hat{q}_2^{(2)}] + d_1[\boldsymbol{P}^{(1)}\hat{q}_1 + \boldsymbol{P}^{(3)}\hat{q}_2^{(2)} + \boldsymbol{P}^{(4)}\hat{q}_2^{(3)}] \\ & + d_{1\Theta}[\boldsymbol{R}^{(1)}\hat{q}_1 + \boldsymbol{R}^{(2)}\hat{q}_2^{(1)} + \boldsymbol{R}^{(4)}\hat{q}_2^{(3)}] + \Theta d_1[\boldsymbol{S}^{(1)}\hat{q}_1 + \boldsymbol{S}^{(3)}\hat{q}_2^{(2)} + \boldsymbol{S}^{(4)}\hat{q}_2^{(3)}] \\ & + \Theta^2 d_1[\boldsymbol{Q}^{(1)}\hat{q}_1 + \boldsymbol{Q}^{(3)}\hat{q}_2^{(2)}]. \end{aligned} \tag{5.16}$$

In obtaining (5.16) we have used the solution (5.14) for $\hat{q}_2$; $\boldsymbol{M}^{(i)}$, $\boldsymbol{N}^{(i)}$, $\boldsymbol{P}^{(i)}$, $\boldsymbol{R}^{(i)}$, $\boldsymbol{S}^{(i)}$, $\boldsymbol{Q}^{(i)}$ for $i = 1,2,3,4$ are 6×6 matrices which need not be given here.

If (5.16) is to have non-trivial solutions, its right-hand side must satisfy an orthogonality condition, which yields an equation for the amplitude $d_1(\Theta)$. Associated with the operator $\hat{\mathbf{C}}_0$ is an adjoint function $\hat{V}$ and the adjoint equation

$$\hat{\mathbf{C}}_0^{\dagger}(\hat{V}) = 0. \tag{5.17a}$$

In (5.17*a*) above, $\hat{\mathbf{C}}_0^{\dagger}$ is given by (4.8) evaluated at k_0, θ_0, i.e.

$$\hat{\mathbf{C}}_0^{\dagger} = \boldsymbol{I}\frac{\partial}{\partial\eta} + \hat{\boldsymbol{A}}_0^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}\frac{\partial}{\partial\tau}. \tag{5.17b}$$

Thus the amplitude equation for d_1 is

$$\boldsymbol{I}_M\, d_{1\Theta\Theta} + \boldsymbol{I}_N\,\Theta d_{1\Theta} + \boldsymbol{I}_P\, d_1 + \boldsymbol{I}_R\, d_{1\Theta} + \boldsymbol{I}_S\,\Theta d_1 + \boldsymbol{I}_Q\,\Theta^2 d_1 = 0, \tag{5.18}$$

where $\boldsymbol{I}_M$, $\boldsymbol{I}_N$, $\boldsymbol{I}_P$, $\boldsymbol{I}_R$, $\boldsymbol{I}_S$, $\boldsymbol{I}_Q$ are double integrals; a typical one is $\boldsymbol{I}_M$ for example, given by

$$\boldsymbol{I}_M = \int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi} \hat{V}^{\mathrm{T}}(\boldsymbol{M}^{(1)}\hat{q}_1 + \boldsymbol{M}^{(2)}\hat{q}_2^{(1)})\,\mathrm{d}\tau\,\mathrm{d}\eta.$$

Equation (5.18) has solutions of the form

$$d_1(\Theta) = \mathrm{e}^{-\hat{\alpha}^2\Theta^2}\hat{F}(\Theta), \tag{5.19}$$

where $\hat{F}(\Theta)$ is a Hermite polynomial and $\hat{\alpha}$ is a suitably chosen real constant. The exponential factor in (5.19) gives the required behaviour as $|\Theta| \to \infty$.

The analysis in this section seems at first to produce solutions that are valid in the complex plane and not on the real axis; this in fact is not true. As we have seen, it is crucial to find the saddle point where T_k, T_θ vanish, even if this means accepting complex values of k and θ. So when such a point is identified, an inner solution in its neighbourhood (note that θ_0 is not located close to the real axis but at $O(1)$ distances from it, as the numerical calculations show) is constructed, which is then matched onto the asymptotic solutions (3.3) valid elsewhere. As was argued earlier, the Stokes constant ζ (cf. (5.9a, b)) connects the solutions on the real axis for $\theta > 0$ and $\theta < 0$. In the numerical search for the saddle point and the subsequent minimization of the Taylor number, the Stokes constant is reduced to zero. Thus, the solution $\boldsymbol{q}_1$ now forms a uniformly valid asymptotic approximation to the true solution on the real axis and is therefore the eigenfunction for the physical problem.

6. Numerical solution

In this section we describe the numerical method used to solve the leading-order eigenvalue problem (4.1) subject to the boundary conditions of no slip at the wall and decay of disturbances at infinity. As described in the previous section, the eigenvalue problem (5.5) must be solved and the values k_0, θ_0 that make T_k, T_θ vanish, identified. Our concern is with neutrally stable modes, so the growth rate σ is identically zero. Thus, for fixed values of k and ξ (note that $\theta = \mathrm{i}\xi$, cf. (5.3), for symmetric modes), the eigenrelation (5.5) is solved to find a value of T. The correct value of T that corresponds to the physically acceptable solutions above which instability occurs, is found when the point (k_0, θ_0), where $T_k = T_\theta = 0$, is located.

The undisturbed flow is periodic in time, so on the basis of Floquet theory we assume a Fourier expansion in time for the disturbances (that is, the disturbances are taken to be periodic in time) and any exponential growth is absorbed into the complex frequency Ω (see (6.1) below); for neutral solutions, however, $\Omega = 0$. So, we expand the leading-order perturbation velocities in the radial and azimuthal directions respectively, in the form

$$(\tilde{u}_1, \tilde{w}_1) = G\,\mathrm{e}^{-\mathrm{i}\Omega\tau} \sum_{n=-\infty}^{\infty} (u_n, w_n)\,\mathrm{e}^{\mathrm{i}n\tau} + \mathrm{C.C.}, \tag{6.1}$$

where the u_n, w_n are functions of η alone. G is a constant and $\Omega = 0$ for neutral modes.

If we now consider the leading-order governing equations (4.1), $\tilde{v}_1$ can be eliminated by use of the continuity equation, and when the pressure $\tilde{p}_1$ is eliminated the following coupled partial differential equations are obtained for $\tilde{u}_1$ and $\tilde{w}_1$:

$$\left(\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2}{\partial\eta^2} - k^2\right)\tilde{u}_1 + 2\mathrm{i}kT_0(k^2 v_{\mathrm{B}0} + v_{\mathrm{B}0\eta\eta})\,\tilde{u}_1 - 2\mathrm{i}kT_0\,v_{\mathrm{B}0}\,\tilde{u}_{1\eta\eta}$$
$$-4k^2 w_{\mathrm{B}0}\,\tilde{w}_1 \cos\theta - 4\mathrm{i}k \sin\theta(w_{\mathrm{B}0\eta}\,\tilde{w}_1 + w_{\mathrm{B}0}\,\tilde{w}_{1\eta}) = 0, \tag{6.2a}$$

$$\left(\frac{\partial^2}{\partial\eta^2} - k^2 - 2\frac{\partial}{\partial\tau}\right)\tilde{w}_1 + 2T_0\,w_{\mathrm{B}0}\,\tilde{u}_1 - 2\mathrm{i}kT_0\,v_{\mathrm{B}0}\,\tilde{w}_1 = 0. \tag{6.2b}$$

Substitution of (6.1) into (6.2a, b) leads to an infinite set of ordinary differential equations for the u_n, w_n obtained at successive powers $\mathrm{e}^{\mathrm{i}n\tau}$. The method of solution is to retain a suitable number of Fourier modes in (6.1), thus reducing the infinite set of equations to a finite one. By a numerical scheme the equations are integrated using as initial condition one of the boundary conditions of the problem; for convenience the condition at 'infinity' is used. A number of independent solutions are needed so that a linear combination of these satisfies the boundary condition of no slip at $\eta = 0$. This yields a finite set of homogeneous algebraic equations which has a non-trivial solution if the determinant of the coefficients of the combination is zero. The evaluation of the determinant provides a value for the function $F(k, \theta, T)$ (cf. (5.5)) which for given values of k, θ and T is generally non-zero. Hence the eigenrelation (5.5) can be satisfied by finding values of T which are roots of the equation (5.5). This is done by a secant method.

The method described above is a modification of the techniques used in relation to boundary-value problems in hydrodynamic stability by, for example, Krueger, Gross & DiPrima (1966) and Seminara & Hall (1976).

Integrations that determine each independent solution are carried out starting at some large value of η, η_∞ say. The value of η_∞ at which we can impose zero velocity on $\tilde{u}_1, \tilde{w}_1$, is in general too large to make it feasible for a modest numerical scheme; this can be avoided by using as initial values at η_∞, solutions for u_n, w_n obtained from (6.2*a*, *b*) after exponentially small terms have been dropped. This technique is discussed in detail by Keller (1968).

7. Results and discussion

Our objective is to find the minimum value of T_0 on the neutral stability curve of T_0 against k_0. With $\Omega = 0$ and k_0, ξ_0 $(\theta = \mathrm{i}\xi_0)$ fixed, an initial guess is made for T_0 and the value of the eigenrelation determinant (see §6) is calculated. A small increment is given to T_0 and the new value of the determinant is found. By use of the secant method the value of T_0 which is a root of the eigenrelation is determined. This is repeated for a range of values of k_0 and ξ_0 in order to locate the points where $\partial T_0/\partial k_0 = \partial T_0/\partial \theta = 0$. The number of Fourier modes retained was a result of numerical experiments, as was the value of η_∞ and the step size. We used two, four or six Fourier modes, and $\eta_\infty = 10$. Forty steps divided the strip $[0, \eta_\infty]$ uniformly, and the independent solutions for each Fourier mode were calculated by use of a fourth-order Runge–Kutta method. For each Fourier mode n, there are three independent solutions at infinity and so the determinant to be calculated has size 39×39. Its value was evaluated by use of NAG subroutines. Convergence of the secant method was very good provided some limitations on the values of ξ_0 were observed. An estimate of these can be obtained by consideration of the fundamental mode in (6.1). For large η, dropping exponentially small terms, the equation for w_0 is, from (6.26),

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - (k_0^2 - \tfrac{1}{2}\mathrm{i}k_0 T_0 \sin\theta)\right] w_0 = 0. \tag{7.1}$$

If $\theta = \mathrm{i}\xi_0$, (7.1) has decaying solutions for large η if and only if

$$\sinh \xi_0 > -\frac{2k_0}{T_0}. \tag{7.2a}$$

At the inner bend, $\theta = \pi + \mathrm{i}\xi_0$, we require

$$\sinh \xi_0 < \frac{2k_0}{T_0}. \tag{7.2b}$$

Initially the neutral stability curves of T_0 versus k_0 were calculated at $\theta = 0$ and $\theta = \pi$ (outer and inner bends respectively) in order to provide an indication as to the position of the onset of instability. The results are given in figure 3. It is seen that the minimum critical Taylor number (for real θ) is lower at the outer bend.

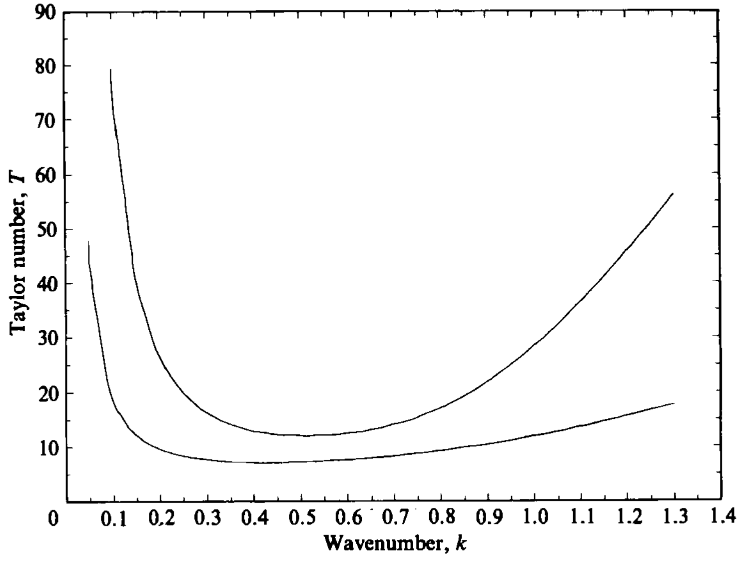

FIGURE 3. The neutral stability curves of T_0 versus k_0 at the inner bend ($\theta = \pi$) and outer bend ($\theta = 0$).

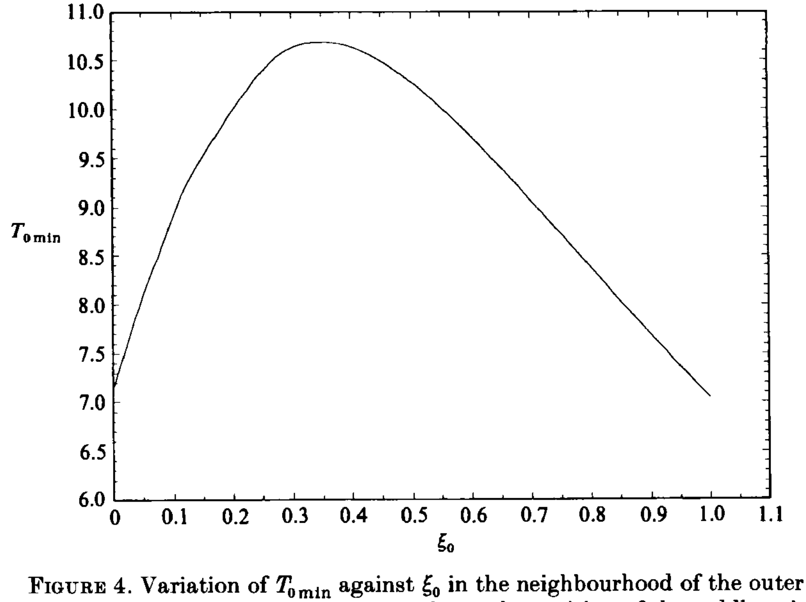

FIGURE 4. Variation of $T_{0\,\mathrm{min}}$ against ξ_0 in the neighbourhood of the outer bend. The turning point at $\xi_0 = 0.3526$ shows the position of the saddle point.

In order to locate the saddle point at the outer bend, the value of ξ_0 was increased, and the minimum Taylor number was found by calculating T_0 for a range of k_0. The value of T_0 was noted, ξ_0 was increased further and the process was continued until T_0 reached a maximum; that is until the point where $\partial T_0/\partial k_0 = \partial T_0/\partial \theta = 0$ was located. The saddle point occurred at $\xi_0 = 0.3526$, and $T_0 = 10.731$, $k_0 = 0.5308$ there. Figure 4 shows the variation of $T_{0\,\mathrm{min}}$ with ξ_0.

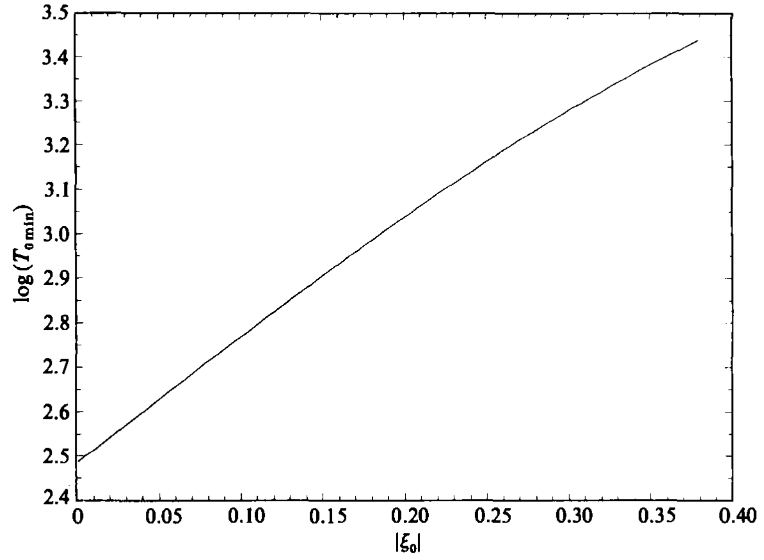

FIGURE 5. Inner bend. Variation of $\log(T_{0\,\min})$ with $\xi_0(\theta = \pi - i\xi_0)$.

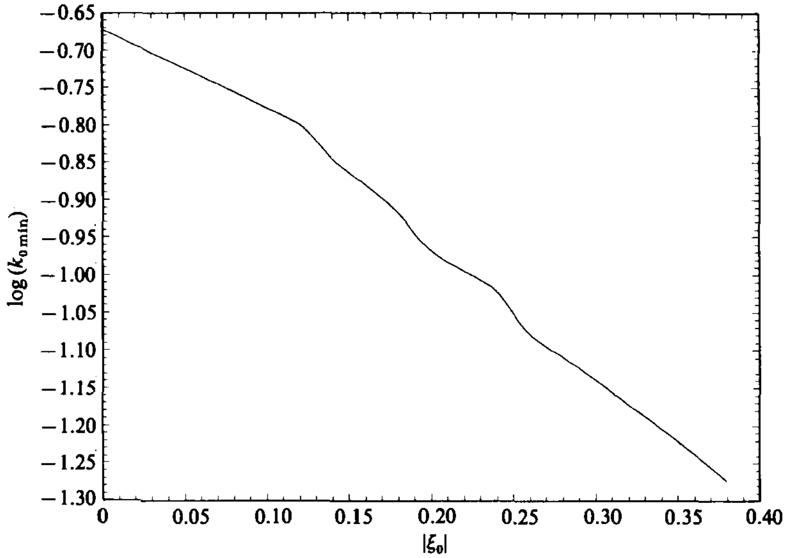

FIGURE 6. Inner bend. Variation of $\log(k_{0\,\min})$ with $\xi_0(\theta = \pi - i\xi_0)$.

To conclude that instability of the form studied here first sets in at the outer bend, it must be shown that the value of T_0 at the saddle point, if it exists, in the vicinity of the inner bend is higher than that at the outer bend. Calculations were carried out at the inner bend, therefore, in an attempt to establish this. These calculations were carried out for negative values of ξ_0, thus satisfying the condition (7.2*b*) automatically. Small enough positive values of ξ_0 also satisfy (7.2*b*) but these were found to produce a decrease in the value of $T_{0\,\min}$, indicating that we should concentrate on the negative axis of ξ_0. The solutions so obtained, however, did not reveal a turning

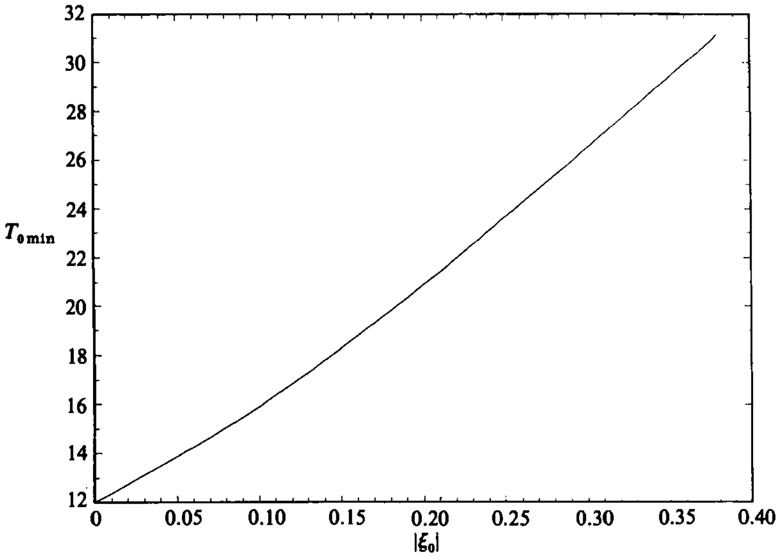

FIGURE 7. Inner bend. Variation of $T_{0\,\mathrm{min}}$ with $\xi_0(\theta = \pi - \mathrm{i}\xi_0)$.

point in the complex plane of θ. Instead, solutions were picked up that had $T_0 \sim \exp(\alpha|\xi_0|)$ as $|\xi_0|$ increased. This is indicated by the log–linear plot of $T_{0\,\mathrm{min}}$ against $|\xi_0|$ in figure 5. The wavenumber on the other hand, decayed exponentially according to $k_0 \sim \exp(-\alpha_2|\xi_0|)$; this is indicated by the log–linear plot of k_0 against $|\xi_0|$ in figure 6 (α_1, α_2 are positive constants). In figure 7 we show the behaviour of $T_{0\,\mathrm{min}}$ with $|\xi_0|$. These results suggest, therefore, that instability of the form considered here is not found at the inner bend.

7.1. *Applications to blood flow*

The problem studied here is of interest in the understanding of the fluid mechanics that operate in the cardiovascular system, where the flow of blood in the large arteries is unsteady and is characterized by large values of R_s. Typically, in the human aorta R_s ranges between 10^3 and 10^4 and in the canine aorta $R_s \approx 4000$. In blood flows like the ones cited above, the observed value of β is small (0.08 for the ascending aorta in humans) and the amplitude of the unsteady oscillatory component of the flow is at least as large as the mean component; this latter fact is an important characterization of blood flow. Our analysis is not inconsistent with these facts. Data for δ give it a value of about 0.2. When the limit $\delta \rightarrow 0$ is taken, the physics of the flow is probably not changed significantly and the study of such limit problems is of value to blood-flow investigations.

We have analysed the stability of Lyne's flow, which has a pressure gradient with zero mean. Usually the pressure gradient, in physiological flows, has a non-zero steady component; this introduces a Dean number, D say, into the problem, defined by $D = (2\delta)^{\frac{1}{2}}G$, where G is the non-oscillatory part of the pressure gradient. For small D, Lyne's flow is the leading-order solution. In blood flow in the canine aorta, for example, $D \approx 2000$ as well as R_s being large. The case $D, R_s \gg 1$, $\beta \ll 1$ is therefore important. To be consistent with blood flow, the amplitude of the unsteady part of the pressure gradient must be larger than the mean steady component. This implies

$W^2\omega/G \gg 1$, or $R_s^{\frac{1}{2}}\beta^{-3} \gg D$ (in our problem $\beta^{-1} \sim R_s$ and so we need $D \ll R_s^{\frac{7}{2}}$). Pedley (1980 §4.2.2) suggests the ordered scaling $1 \ll \beta^{-1} \ll D < R_s$ in view of observational data. This ordering is violated in our problem, since $R_s \sim \beta^{-1}$. Blennerhasset (1976), however, took the limit $\beta \rightarrow 0$ and looked at the cases $D \leqslant R_s$ which make the problem physiologically reasonable. The flow now has a Stokes layer on the tube walls and the solutions inside the Stokes layer are those given by Lyne's leading-order terms and whose linear stability we have calculated here. As an aside, we mention that the physically important case occurs when the direction of the centrifuging (secondary steady streaming) changes from being outwards to inwards. This transition occurs when $R_s = O(D^{\frac{1}{3}})$ (found when contributions to the secondary streaming due to a steady pressure gradient balance those due to the flow driven by a Stokes layer with zero mean pressure gradient). The reason for the importance of such a regime is that from a physiological point of view we need to predict the wall shear stresses and the axial velocity in the core. Smith (1975) has studied this and a number of other limits theoretically by means of asymptotic techniques.

If we use the data for the canine aorta to calculate the Taylor number for the flow, we find it to be approximately 165. This implies that the flow is linearly unstable, at the outer bend, to disturbances of the type considered here. Observations on blood flow are not conclusive as to whether the flow is laminar or turbulent, but varied flow regimes are reported instead. It is not a straightforward matter to make a direct correlation between atherogenesis (deposition of fatty substances on arterial walls) and the fluid dynamics, due to the complicating biochemical factors present. There is evidence, however, that links the distribution of fatty streaks with shear stress distributions at the vessel walls.

For example, it is observed that in normal diets fatty streaks are usually deposited at the inner walls of curved arteries. A reason attributed to this, which is consistent with our results, is that due to the high shear at the outer bend (unstable flow here) the permeability of the vessel walls increases and fatty molecules have a chance of being diffused out of the plasma. At the inner bend, however, this is less likely to happen, and deposition takes place. There are other factors to be considered before this theory is rendered complete, and for a full discussion on atherogenesis the reader is referred to Pedley (1980, chapter 1) and Lighthill (1975, chapter 13).

The author is grateful to Professor P. Hall both for suggesting this problem and for his continual guidance throughout the course of this work. Thanks are also due to the Science and Engineering Research Council for a support grant and to the Courant Institute where the final draft of the paper was prepared.

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Appendix

The functions $J(\theta)$, $H(\theta)$ in the amplitude equation (4.9) are given by

$$J(\theta) = \int_{\eta=0}^{\infty}\int_{\tau=0}^{2\pi}\{V_1[-\tfrac{1}{2}\tilde{v}_{1\eta} + (T_0 v_{B0} - ik)\,\tilde{u}_1] + V_2[2\tilde{p}_1 + 2(T_0 v_{B0} - ik)\,\tilde{v}_1] + 2V_3[T_0 v_{B0} - ik]\,\tilde{w}_1 + V_4\tilde{v}_1\}\,d\eta\,d\tau, \quad (A\,1)$$

$$H(\theta) = \int_{\eta=0}^{\infty} \int_{\tau=0}^{2\pi} \{ V_1[-\tfrac{1}{2}\tilde{v}_{1\eta\theta} + \mathrm{i}k(T_0 v_{\mathrm{B}1} + T_1 v_{\mathrm{B}0})\tilde{u}_1 - T_0 u_{\mathrm{B}0\eta}\tilde{u}_1$$
$$+ (T_0 v_{\mathrm{B}0} - \mathrm{i}k)\tilde{u}_{1\theta} + (\tfrac{1}{2}\mathrm{i}k - \mathrm{i}kT_0 u_{\mathrm{B}0} - 2T_0 v_{\mathrm{B}0})\tilde{v}_1 - 2w_{\mathrm{B}1}\tilde{w}_1 \cos\theta]$$
$$+ V_2[2\tilde{p}_{1\theta} + (1 - 2T_0 u_{\mathrm{B}0})\tilde{v}_{1\eta} + (2T_0 v_{\mathrm{B}0} - 2T_0 v_{\mathrm{B}1\eta} - 2T_1 v_{\mathrm{B}0\eta} - \mathrm{i}k)\tilde{u}_1$$
$$+ \left(2\mathrm{i}kT_0 v_{\mathrm{B}1} + 2\mathrm{i}kT_1 v_{\mathrm{B}0} + 2T_0 v_{\mathrm{B}0\theta} - \mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}\theta}\right)\tilde{v}_1 + 2(T_0 v_{\mathrm{B}0} - \mathrm{i}k)\tilde{v}_{1\theta}$$
$$+ 4w_{\mathrm{B}1}\tilde{w}_1 \sin\theta] + V_3[(1 - 2T_0 u_{\mathrm{B}0})\tilde{w}_{1\eta} - 2(T_0 w_{\mathrm{B}1\eta} + T_1 w_{\mathrm{B}0\eta})\tilde{u}_1$$
$$+ 2\mathrm{i}k(T_1 v_{\mathrm{B}0} + T_0 v_{\mathrm{B}1})\tilde{w}_1 + 2(T_0 v_{\mathrm{B}0} - \mathrm{i}k)\tilde{w}_{1\theta}] + V_4[\tilde{u}_1 + \tilde{v}_{1\theta}]\}\,\mathrm{d}\eta\,\mathrm{d}\tau. \qquad (\mathrm{A}\ 2)$$

Thus, the expansions for small θ are

$$J(\theta) = J_0 + \theta^{\frac{1}{2}} J_1 + \dots, \qquad (\mathrm{A}\ 1a)$$

$$H(\theta) = \theta^{-\frac{1}{2}} H_0 + O(1), \qquad (\mathrm{A}\ 2a)$$

where J_0, J_1, H_0 can be found by substituting (A 1a), (A 2a) into (A 1), (A 2) and picking out powers of $\theta^{\frac{1}{2}}$.

The 6×6 matrix $\boldsymbol{L}$ in equation (4.2) has non-zero elements given by, using the usual matrix notation,

$$L_{12} = -\frac{1}{2}\left(b_1 \frac{\partial}{\partial\theta} + \frac{\mathrm{d}b_1}{\mathrm{d}\theta}\right),$$

$$L_{14} = \mathrm{i}k(T_0 v_{\mathrm{B}1} + T_1 v_{\mathrm{B}0})b_1 - T_0 u_{\mathrm{B}0\eta} b_1 + b_1(T_0 v_{\mathrm{B}0} - \mathrm{i}k)\frac{\partial}{\partial\theta} + (T_0 v_{\mathrm{B}0} - \mathrm{i}k)\frac{\mathrm{d}b_1}{\mathrm{d}\theta},$$

$$L_{15} = (\tfrac{1}{2}\mathrm{i}k - \mathrm{i}kT_0 v_{\mathrm{B}0} - 2T_0 v_{\mathrm{B}0})b_1,$$

$$L_{16} = -2b_1 w_{\mathrm{B}1} \cos\theta,$$

$$L_{21} = 2\left(b_1 \frac{\partial}{\partial\theta} + \frac{\mathrm{d}b_1}{\mathrm{d}\theta}\right),$$

$$L_{22} = b_1(1 - 2T_0 u_{\mathrm{B}0}),$$

$$L_{24} = b_1(2T_0 v_{\mathrm{B}0} - 2T_0 v_{\mathrm{B}1\eta} - 2T_1 v_{\mathrm{B}0\eta} - \mathrm{i}k),$$

$$L_{25} = b_1\left(2\mathrm{i}kT_0 v_{\mathrm{B}1} + 2\mathrm{i}kT_1 v_{\mathrm{B}0} + 2T_0 v_{\mathrm{B}0\theta} - \mathrm{i}\frac{\mathrm{d}k}{\mathrm{d}\theta} + 2T_0 v_{\mathrm{B}0}\frac{\partial}{\partial\theta} - 2\mathrm{i}k\frac{\partial}{\partial\theta}\right) + 2\frac{\mathrm{d}b_1}{\mathrm{d}\theta}(T_0 v_{\mathrm{B}0} - \mathrm{i}k),$$

$$L_{26} = 4b_1 w_{\mathrm{B}1} \sin\theta,$$

$$L_{33} = b_1(1 - 2T_0 u_{\mathrm{B}0}),$$

$$L_{34} = -2b_1(T_0 w_{\mathrm{B}1\eta} + T_1 w_{\mathrm{B}0\eta}),$$

$$L_{36} = 2b_1\left(\mathrm{i}kT_1 v_{\mathrm{B}0} + \mathrm{i}kT_0 v_{\mathrm{B}1} + T_0 v_{\mathrm{B}0}\frac{\partial}{\partial\theta} - \mathrm{i}k\frac{\partial}{\partial\theta}\right) + 2\frac{\mathrm{d}b_1}{\mathrm{d}\theta}(T_0 v_{\mathrm{B}0} - \mathrm{i}k),$$

$$L_{44} = b_1,$$

$$L_{45} = b_1 \frac{\partial}{\partial\theta} + \frac{\mathrm{d}b_1}{\mathrm{d}\theta}.$$

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